Projective superspace and hyperkähler sigma models on cotangent bundles of Hermitian symmetric spaces

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Abstract. We review the projective-superspace construction of four-dimensional $\mathscr{N}=2$ supersymmetric sigma models on (co)tangent bundles of the classical Hermitian symmetric spaces.

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Supersymmetry in sigma models is closely related to the geometry of target space [1]. In $\mathscr{N}=1$ models in four space-time dimensions, the target space manifold must be Kähler [1]. General $\mathscr{N}=1$ sigma models can be written in terms of a Kähler potential depending on $\mathscr{N}=1$ chiral superfields and their conjugates. 4D $\mathscr{N}=2$ models require the target space geometry to be hyperkähler [2]. Hyperkähler metrics are difficult to construct explicitly. Thus, manifest $\mathscr{N}=2$ formulations are needed in order to generate $\mathscr{N}=2$ sigma models and, therefore, hyperkähler metrics. Projective superspace [3, 4] provides such a formulation, and it is thus suitable for the construction.

Projective superspace formulation has led to the discovery of several multiplets that can be used to construct new hyperkähler metrics. One of the most interesting projective multiplets is the so-called polar multiplet [5, 6] (see also [7]), for it can be used to describe a charged $U(1)$ hypermultiplet coupled to a vector multiplet [6], and therefore is analogous to the $\mathscr{N}=1$ chiral superfield. The polar multiplet is described by an arctic superfield $\Upsilon(\zeta)$ and its complex conjugate composed with the antipodal map $\bar{\zeta} \to -1/\zeta$, the antarctic superfield $\breve{\Upsilon}(\zeta)$. It is required to possess certain holomorphy properties on a punctured two-plane parametrized by the complex variable ζ (the latter may be interpreted as a projective coordinate on $\mathbf{C}P^1$). When realized in ordinary $\mathscr{N}=1$ superspace, $\Upsilon(\zeta)$ and $\breve{\Upsilon}(\zeta)$ are generated by an infinite set of ordinary superfields:

$$\Upsilon(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma\zeta + O(\zeta^2)~, \qquad \breve{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\zeta)^{-n}~. \tag{1}$$

Here Φ is chiral, Σ complex linear,

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ , \qquad\qquad \bar{D}^2\Sigma = 0 \ , \tag{2}$$

and the remaining component superfields are unconstrained complex superfields. Sigma model couplings of several polar multiplets are described by an action of the form [5]

$$S[\Upsilon,\breve{\Upsilon}] = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^8 z\, \mathscr{K}\big(\Upsilon(\zeta), \breve{\Upsilon}(\zeta), \zeta\big) \ . \tag{3}$$

It was observed in [8, 9] that there exists a large subclass in the family of models (3) with interesting geometric properties. It corresponds to the case when the Lagrangian $\mathscr{K}$ in (3) has no explicit ζ-dependence,

$$S[\Upsilon,\breve{\Upsilon}] = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^8 z\, K\big(\Upsilon^I(\zeta), \breve{\Upsilon}^{\bar{I}}(\zeta)\big) \ , \tag{4}$$

and then K can be interpreted as the Kähler potential of a Kähler manifold $\mathscr{M}$. Such a theory occurs as a minimal $\mathscr{N}=2$ extension of the general four-dimensional $\mathscr{N}=1$ supersymmetric nonlinear sigma model [1]

$$S[\Phi,\bar{\Phi}] = \int \mathrm{d}^8 z\, K(\Phi^I, \bar{\Phi}^{\bar{J}}) \ . \tag{5}$$

The target space of the $\mathscr{N}=2$ sigma model (4) turns out to be (an open domain of the zero section) of the cotangent bundle of $\mathscr{M}$ [9]. By construction, the model (4) involves an infinite set of auxiliary superfields. The hard technical problem is to eliminate these auxiliaries, and then dualize the complex linear superfields Σ's into chiral ones, in accordance with the generalized Legendre transform [5]. This problem was solved in [9, 10] for the case $\mathscr{M} = \mathbf{C}P^n$, the complex projective space, and incomplete results were also obtained in [10] for the complex quadric $SO(n+2)/SO(n)\times SO(2)$. Recently we have solved the problem for a large class of classical Hermitian symmetric spaces [11]. In what follows, a brief review of our construction is presented.

The extended supersymmetric sigma model (4) inherits all the geometric features of its $\mathscr{N}=1$ predecessor (5). The Kähler invariance of the latter, $K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})$ turns into $K(\Upsilon,\breve{\Upsilon}) \to K(\Upsilon,\breve{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\breve{\Upsilon})$ for the model (4). A holomorphic reparametrization of the Kähler manifold, $\Phi^I \to f^I\big(\Phi\big)$, has the following counterpart $\Upsilon^I(\zeta) \to f^I\big(\Upsilon(\zeta)\big)$ in the $\mathscr{N}=2$ case. Therefore, the physical superfields of the $\mathscr{N}=2$ theory

$$\Upsilon^I(\zeta)\Big|_{\zeta=0} \ = \ \Phi^I \ , \qquad\qquad \frac{\mathrm{d}\Upsilon^I(\zeta)}{\mathrm{d}\zeta}\Big|_{\zeta=0} \ = \ \Sigma^I \ , \tag{6}$$

should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point. Thus the variables (Φ^I,Σ^J) parametrize the tangent bundle $T\mathscr{M}$ of the Kähler manifold $\mathscr{M}$ [8].

The presence of auxiliary superfields $\Upsilon_2, \ \Upsilon_3 \ldots$ in (1) makes $\mathscr{N}=2$ supersymmetry manifest, but the physical content of the theory is hidden. To describe the theory in

terms of the physical superfields Φ and Σ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{\mathrm{d}\zeta}{\zeta}\,\zeta^n\,\frac{\partial K(\Upsilon,\breve{\Upsilon})}{\partial \Upsilon^I} \;=\; \oint \frac{\mathrm{d}\zeta}{\zeta}\,\zeta^{-n}\,\frac{\partial K(\Upsilon,\breve{\Upsilon})}{\partial \breve{\Upsilon}^{\bar{I}}} \;=\; 0\;, \qquad n\geq 2\,. \tag{7}$$

In general, these equations can be solved only pertubatively. However, as outlined in [9] and elaborated in detail in [11], the auxiliary fields may be eliminated exactly for any Hermitian symmetric space $\mathscr{M}$. Let us sketch the procedure of constructing such a solution $\Upsilon_*(\zeta)\equiv\Upsilon_*(\zeta;\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})$ under the initial conditions (6).

Given an arbitrary point $p_0\in\mathscr{M}$, there exists a Kähler normal coordinate frame with origin at p_0 such that

$$K_{I_1\dots I_m\bar{J}_1\dots\bar{J}_n}\Big|_{\Phi=0}=0\;, \qquad m\neq n \tag{8}$$

and therefore $K(\Phi,\bar{\Phi})=F(\Phi\bar{\Phi})$, with $F(\Phi\bar{\Phi})$ a real analytic function. Now, for any tangent vector Σ_0 at the origin, one can see that

$$\Upsilon_0(\zeta)=\zeta\Sigma_0, \qquad \breve{\Upsilon}_0(\zeta)=-\frac{\bar{\Sigma}_0}{\zeta} \tag{9}$$

solve the equations (7) at $\Phi=0$. We set $\Upsilon_*(\zeta;\Phi=0,\bar{\Phi}=0,\Sigma_0,\bar{\Sigma}_0)=\zeta\Sigma_0$. A next step is to distribute this solution to any point Φ of the manifold $\mathscr{M}$, that is to make use of $\Upsilon_*(\zeta;\Phi=0,\bar{\Phi}=0,\Sigma_0,\bar{\Sigma}_0)$ in order to obtain $\Upsilon_*(\zeta;\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})$.

Let G be the isometry group of $\mathscr{M}$. It acts transitively on $\mathscr{M}$ by holomorphic transformations. Let U be the open domain on which the normal coordinate system is defined. It can always be chosen such that we can construct a coset representative, $\mathscr{S}\colon U\to G$, where $\mathscr{S}(p)\colon \mathscr{M}\to\mathscr{M}$ is a holomorphic isometry transformation with the property

$$\mathscr{S}(p)\,p_0=p\;, \qquad \mathscr{S}(p)\in G\;, \qquad \forall p\in U\;.$$

In other words, $\mathscr{S}(p)$ maps the origin to p. In local coordinates, $\mathscr{S}(p)=\mathscr{S}(\Phi,\bar{\Phi})$, and it acts on a generic point $q\in U$ parametrized by complex variables $(\Psi^I,\bar{\Psi}^{\bar{J}})$ as follows:

$$\Psi\to\Psi'=f(\Psi;\Phi,\bar{\Phi})\;, \qquad f(0;\Phi,\bar{\Phi})=\Phi\;. \tag{10}$$

It is crucial that the holomorphic isometry transformations leave the equations (7) invariant. This means that applying $\mathscr{S}(\Phi,\bar{\Phi})$ to $\Upsilon_0(\zeta)$, eq. (9), gives

$$\Upsilon_0(\zeta)\;\to\;\Upsilon_*(\zeta)=f(\Upsilon_0(\zeta);\Phi,\bar{\Phi})=f(\Sigma_0\,\zeta;\Phi,\bar{\Phi})\;, \qquad \Upsilon_*(0)=\Phi\;. \tag{11}$$

Imposing the second initial condition in (6),

$$\Sigma^I=\Sigma_0^J\,\frac{\partial}{\partial\Psi^J}\,f^I(\Psi;\Phi,\bar{\Phi})\Big|_{\Psi=0}\;, \tag{12}$$

we are in a position to uniquely express Σ_0 in terms of Σ and Φ, $\bar{\Phi}$. By construction, Σ is a complex linear superfield constrained as in (2). As to Σ_0, it obeys a generalized linear constraint that follows from (12) by requiring $\bar{D}^2\Sigma=0$.

Once the solution $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta;\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})$ is given, we are in a position in principle to compute the tangent bundle action

$$S_{\rm tb}[\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}] = \frac{1}{2\pi {\rm i}} \oint \frac{{\rm d}\zeta}{\zeta} \int {\rm d}^8 z\, K\big(\Upsilon_*(\zeta),\breve{\Upsilon}_*(\zeta)\big)\ . \tag{13}$$

But it is extremely important to choose a simplest coset representative (using the natural freedom in its choice $\mathscr{S}(\Phi,\bar{\Phi}) \to \mathscr{S}(\Phi,\bar{\Phi})\,h(\Phi,\bar{\Phi})$, with $h(\Phi,\bar{\Phi})$ taking its values in the isotropy subgroup H at p_0). With a complicated coset representative chosen, it will be practically impossible to do the contour integral on the right of (13). Finally, after having evaluated the contour integral in (13), it only remains to dualize the complex linear tangent variables Σ's into chiral one-forms, in accordance with the generalized Legendre transform [5]. The target space for the model obtained is $T^*\mathscr{M}$ in the compact case, and a part of $T^*\mathscr{M}$ in the non-compact case. In our work [11], the above procedure was carried out for the following Hermitian symmetric spaces (HSS):

compact HSS	$\frac{U(n+m)}{U(n)\times U(m)}$	$\frac{SO(2n)}{U(n)}$	$\frac{Sp(n)}{U(n)}$	$\frac{SO(n+2)}{SO(n)\times SO(2)}$
non-compact HSS	$\frac{U(n,m)}{U(n)\times U(m)}$	$\frac{SO^*(2n)}{U(n)}$	$\frac{Sp(n,\mathbf{R})}{U(n)}$	$\frac{SO_0(n,2)}{SO(n)\times SO(2)}$

As an example, consider the Grassmannian $\mathscr{M} = G_{m,n+m}$. Its Kähler potential is

$$K(\Phi,\Phi^\dagger) = \ln\det(\mathbf{1}_m + \Phi^\dagger\Phi) = \ln\det(\mathbf{1}_n + \Phi\Phi^\dagger)\ . \tag{14}$$

where $\Phi = (\Phi^{i\alpha})(i=1,\dots,n,\ \alpha=1,\dots,m)$. The useful coset representative is

$$\mathscr{S}(\Phi,\bar{\Phi}) = \begin{pmatrix} \underline{s} & \Phi s \\ -\Phi^\dagger \underline{s} & s \end{pmatrix}\ , \qquad s^{-2} = \Phi^\dagger\Phi + \mathbf{1}_m\ , \quad \underline{s}^{-2} = \Phi\Phi^\dagger + \mathbf{1}_n\ . \tag{15}$$

Its use leads to the tangent bundle action [11]

$$S_{\rm tb} = \int {\rm d}^8 z\, \Big\{K(\Phi,\Phi^\dagger) + \ln\det\Big(\mathbf{1}_m - (\mathbf{1}_m + \Phi^\dagger\Phi)^{-1}\Sigma^\dagger(\mathbf{1}_n + \Phi\Phi^\dagger)^{-1}\Sigma\Big)\Big\}\ . \tag{16}$$

Finally, applying the generalized Legendre transform to (16) gives the $\mathscr{N}=2$ sigma model originally constructed in [12].

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